Euler Parameters and Bowling ball dynamics a la Huston et al.

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Definition

Simple rotation of Θ about a unit vector $\hat{\lambda}$

$$\varepsilon_i \equiv \lambda_i \sin\frac{\theta}{2}, i = 1,2,3$$

$$\varepsilon_4 \equiv \cos\frac{\theta}{2}$$

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

Relation to direction cosine matrix

$$
{}^a\mathbf{C}^b = \begin{bmatrix} -2\,\epsilon_2{}^2 - 2\,\epsilon_3{}^2 + 1 & 2\,\epsilon_1\,\epsilon_2 - 2\,\epsilon_3\,\epsilon_4 & 2\,\epsilon_1\,\epsilon_3 + 2\,\epsilon_2\,\epsilon_4 \\ 2\,\epsilon_1\,\epsilon_2 + 2\,\epsilon_3\,\epsilon_4 & -2\,\epsilon_1{}^2 - 2\,\epsilon_3{}^2 + 1 & 2\,\epsilon_2\,\epsilon_3 - 2\,\epsilon_1\,\epsilon_4 \\ 2\,\epsilon_1\,\epsilon_3 - 2\,\epsilon_2\,\epsilon_4 & 2\,\epsilon_1\,\epsilon_4 + 2\,\epsilon_2\,\epsilon_3 & -2\,\epsilon_1{}^2 - 2\,\epsilon_2{}^2 + 1 \end{bmatrix}
$$

Using `dircos`

```
   (1) %DIRCOS - Euler
   (2) frames a,b
   (3) variables e{4}
   (4) dircos(a,b,euler,e1,e2,e3,e4)
-> (5) a_b[1,1] = 1 - 2*e2^2 - 2*e3^2
-> (6) a_b[1,2] = 2*e1*e2 - 2*e3*e4
-> (7) a_b[1,3] = 2*e1*e3 + 2*e2*e4
-> (8) a_b[2,1] = 2*e1*e2 + 2*e3*e4
-> (9) a_b[2,2] = 1 - 2*e1^2 - 2*e3^2
-> (10) a_b[2,3] = 2*e2*e3 - 2*e1*e4
-> (11) a_b[3,1] = 2*e1*e3 - 2*e2*e4
-> (12) a_b[3,2] = 2*e1*e4 + 2*e2*e3
-> (13) a_b[3,3] = 1 - 2*e1^2 - 2*e2^2
```

Derivatives

$$\omega \equiv [\omega_1 \quad \omega_2 \quad \omega_3 \quad 0]$$

$$\varepsilon \equiv [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \varepsilon_4]$$

$$E = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

$$\dot{\varepsilon} = \frac{1}{2}\omega E^T$$

$$\omega = 2\dot{\varepsilon}E$$

Derivation is in Kane, Likins, Levinson (1993). Spacecraft Dynamics.

Alternate forms of ω?

Kane et al.

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = 2\cdot\begin{bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{\varepsilon}_3\\ \dot{\varepsilon}_4\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1\\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2\\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3\\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4\end{bmatrix}$$

Huston et al.

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = 2\cdot\begin{bmatrix}\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & -\varepsilon_1\\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & -\varepsilon_2\\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & -\varepsilon_3\\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4\end{bmatrix}\cdot\begin{bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{\varepsilon}_3\\ \dot{\varepsilon}_4\end{bmatrix}$$

From Kane, Likins, Levinson (1993). Spacecraft Dynamics.
And Huston, Passerello, Winget, & Sears (1979). On the dynamics of a weighted bowling ball.

Autolev implementation

```
%     Newtonian, bodies, frames
newtonian A
bodies B
%     Variables, constants, points
variables q{7}',u{7}'...
%     KDEs
q1' = u1 ...
%     Position vectors ...
%     Euler parameters
dircos(A,B,Euler,e1,e2,e3,e4)
%     Velocities
%     Motion constraint
%     KDEs
kindiffs(A,B,Euler,e1,e2,e3,e4)
%     Accelerations ...
%     Forces ...
%     Equations of motion, output Matlab file...
```

Overview

CM Offset: ✓

Distinct moment of inertia: 3

Friction: Coulomb

Oil pattern: uniform

Newton-Euler / d'Alembert

Euler Parameters

Notation

A lane

B ball

C contact point

G center of mass

Q geometric center

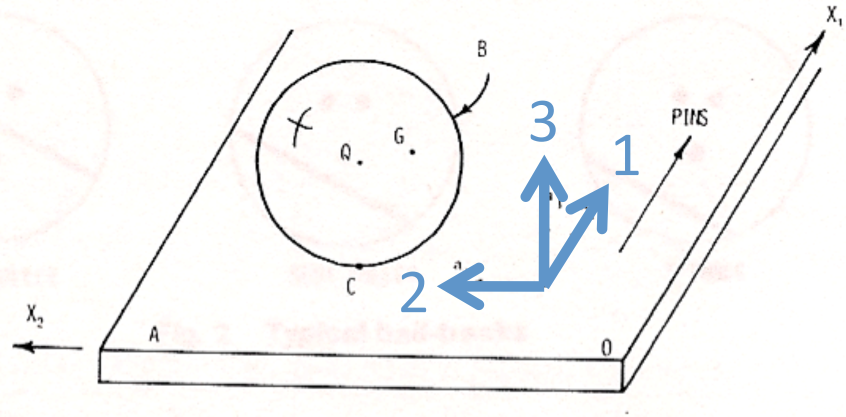


Fig. 1 Bowling ball and lane

Kinematics

$$\omega = \omega_1 \hat{a}_1 + \omega_2 \hat{a}_2 + \omega_3 \hat{a}_3$$

$$\alpha = \dot{\omega}_1 \hat{a}_1 + \dot{\omega}_2 \hat{a}_2 + \dot{\omega}_3 \hat{a}_3$$

Given ω, the orientation ε can be found?

Kinematics (2)

C contact point
G center of mass
Q geometric center

Defining velocity and acceleration of geometric center

$$\mathbf{v}_Q = \dot{x}_1\mathbf{a}_1 + \dot{x}_2\mathbf{a}_2$$

$$\mathbf{a}_Q = \ddot{x}_1\mathbf{a}_1 + \ddot{x}_2\mathbf{a}_2$$

Velocity and acceleration of center of mass – Q and G are on a rigid body

$$\mathbf{V}_G = \mathbf{V}_Q + \boldsymbol{\omega} \times \mathbf{P}$$

$$\mathbf{a}_G = \mathbf{a}_Q + \alpha \times \mathbf{P} + \omega \times (\omega \times \mathbf{P})$$

Velocity of contact point

$$\mathbf{V}_C = \mathbf{V}_Q + \boldsymbol{\omega} \times (-r\mathbf{a}_3)$$

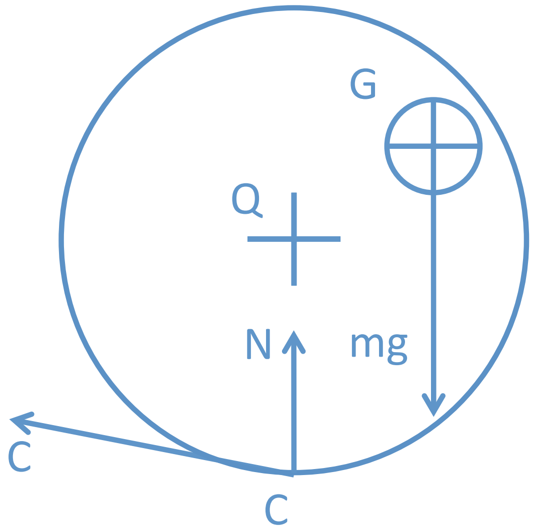
Slip - stick

Rolling

$$v_C = r\omega_2 \hat{a}_1 - r\omega_1 \hat{a}_2$$

Note that ω_3 is a 3rd degree of freedom, but does not affect the motion

Kinetics



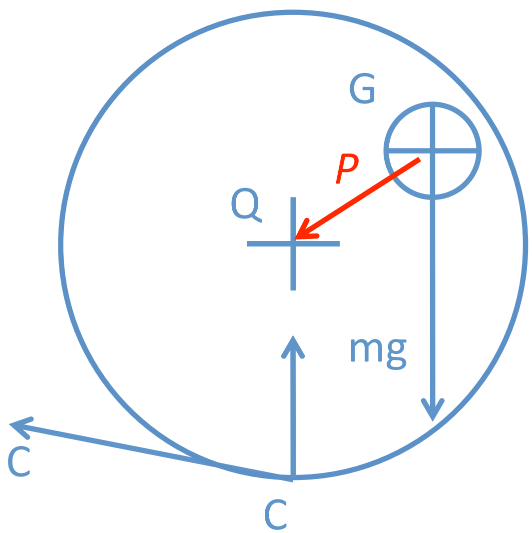
$$\tau = \frac{{}^{A}v^{C}}{\left| {}^{A}v^{C} \right|}$$

Contact force $C = -\mu N\tau + Na_3$

Inertia force $F = -ma_G$

Inertia torque $T = -I\alpha - \omega \times I \cdot \omega$

Governing equations



$$F + C - mg\hat{a}_3 = 0$$

$$T - P \times m(ga_3 + a_G) - r\hat{a}_3 \times C = 0$$

Rolling

$$T - (r\hat{a}_3 + P) \times m(ga_3 + a_G) = 0$$

Parameters and initial conditions

r=0.109 m

mg=71.32 N (16 lbs)

$|\omega|$=10 rad/s

$\omega_1=\omega_2$

$\omega_3=0$

Results: hook and ω

Definition of percent hook:

$$H = \frac{100\Delta x_2}{18.288}$$

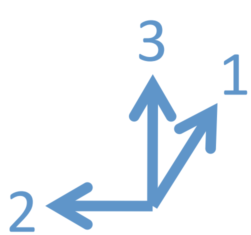


Table 1 Effect of initial angular velocity direction on hook

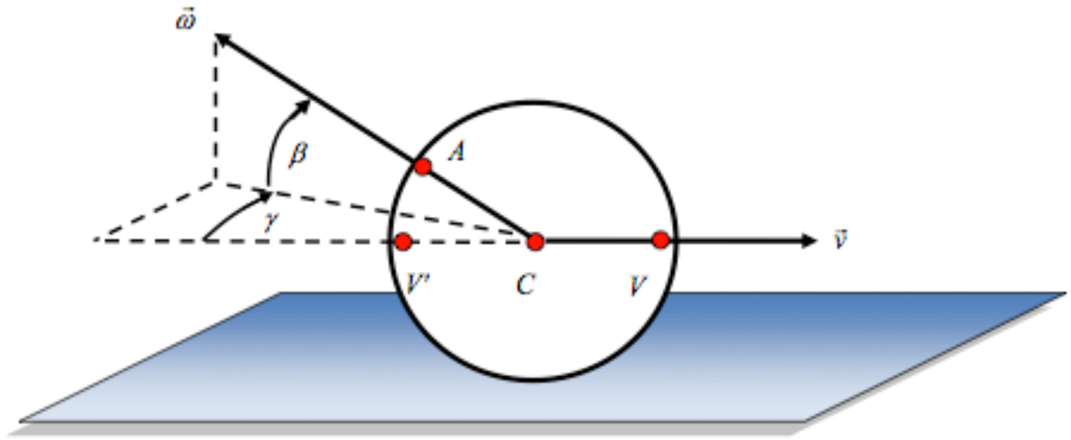
INITIAL ANGULAR VELOCITY (RAD/SEC)			PERCENT HOOK
ω_1	ω_2	ω_3	H
0.0	10.0	0.0	0.0
-2.588	9.659	0.0	0.34
-5.0	8.66	0.0	0.64
-7.07	7.07	0.0	0.87
-9.659	2.588	0.0	1.10
-10.0	0.0	0.0	1.09
-6.96	6.96	1.736	0.86
-6.64	6.64	3.42	0.82
-6.12	6.12	5.0	0.75
-7.07	7.07	0.25	0.87
-7.07	7.07	0.50	0.87
-7.07	7.07	1.0	0.87
-7.07	7.07	2.0	0.87
-7.07	7.07	5.0	0.87

Results: hook and ω (2)

Axis rotation γ and axis tilt β

$$\gamma = \tan^{-1}\left(\frac{\omega_2}{-\omega_1}\right)$$

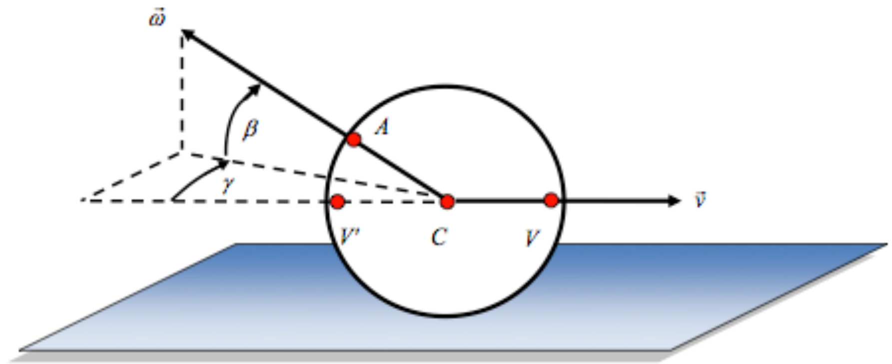
$$\beta = \tan^{-1}\left(\frac{\omega_3}{\sqrt{\omega_1^2 + \omega_2^2}}\right)$$



From King, Perkins, et al. (2010). Bowling dynamics revealed by miniature wireless MEMS IMU.

Results: hook and ω (3)

γ	β	|ω|	H
90	0	10	0.00
75	0	10	0.34
60	0	10	0.64
45	0	10	0.87
15	0	10	1.10
0	0	10	1.09
45	10	10	0.86
45	20	10	0.82
45	30	10	0.75
45	1.43	10	0.87
45	2.86	10.01	0.87
45	5.71	10.05	0.87
45	11.3	10.20	0.87
45	26.6	11.18	0.87



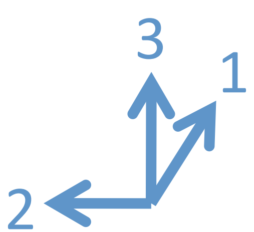
Hook ↑ with γ ↓, for a constant |ω|

Hook ↓ with β ↑, for a constant |ω|

Hook constant with ω_3 ↑

Results: hook and CM offset

CM offsets defined in the A frame



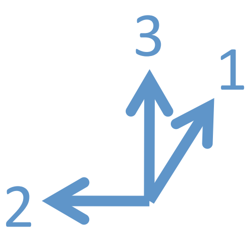
MASS CENTER POSITION (mm)			PERCENT HOOK
P_1	P_2	P_3	H
1.076	-1.076	0.0	0.49
-1.076	1.076	0.0	1.16
0.536	1.426	0.0	1.16
-1.426	0.536	0.0	1.15
-0.878	0.878	0.878	1.22
-0.878	0.878	-0.878	1.00
0.878	-0.878	0.878	0.69
0.878	-0.878	-0.878	0.44
1.524	0.0	0.0	0.62
-1.524	0.0	0.0	1.08
0.0	1.524	0.0	1.10

Results: hook and I

I defined in the A frame

I_{11} ↑, H ↑

I_{22} ↑, H ↓



INERTIA TENSOR I_{ij} ($kg\ m^2$)	PERCENT HOOK $\underline{H}$
$\begin{bmatrix} 0.0347 & 0.0 & 0.0 \\ 0.0 & 0.0347 & 0.0 \\ 0.0 & 0.0 & 0.0347 \end{bmatrix}$	0.872
$\begin{bmatrix} 0.0347 & -0.0135 & 0.0 \\ -0.0135 & 0.0347 & 0.0 \\ 0.0 & 0.0 & 0.0256 \end{bmatrix}$	0.873
$\begin{bmatrix} 0.0347 & 0.0 & 0.0 \\ 0.0 & 0.0347 & -0.0135 \\ 0.0 & -0.0135 & 0.0347 \end{bmatrix}$	0.790
$\begin{bmatrix} 0.0483 & 0.0 & 0.0 \\ 0.0 & 0.0347 & 0.0 \\ 0.0 & 0.0 & 0.0347 \end{bmatrix}$	0.96
$\begin{bmatrix} 0.0347 & 0.0 & 0.0 \\ 0.0 & 0.0483 & 0.0 \\ 0.0 & 0.0 & 0.0347 \end{bmatrix}$	0.718